Statistical Inference-II

Likelihood ratio test

Obtain the likelihood function from the given density. When H_o: $\mu = \mu_o$ obtain $L(\omega)$ by putting $\mu = \mu_0$ in L(X). the estimate the parameter of the distribution and replace it into $L(\omega)$ then it become $L(\hat{\omega})$ and when H_A : $\mu \neq \mu_o$ then estimate the parameter from $L(\underline{X})$ by using MLE method . Then put these estimators in L(X) which will become $L(\hat{\Omega})$.

Then likelihood ratio test.

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \qquad \text{or} \qquad \frac{L(\hat{\omega})}{L(\hat{\Omega})} \leq k_\alpha$$

The estimator which is use to estimate the parameters is only used when they are unknown. Otherwise if the parameters of the distribution are known then there no need to estimate the parameter. The method of estimation used will be MLE method.

Q.No .1

Derive likelihood ratio test based on n independent observation for $H_0 : \theta = 1$ Vs $H_A : \theta \neq 1$

For

$$f(x) = C\ e^{-\theta x}\ x^2 \qquad x \geq 0$$

Solution

First we find the value of "c"

As we known that

$$Area = \int_{-\infty}^{\infty} f(x)\ \ d(x)$$

$$1 = \int_0^{\infty} C e^{-\theta x} x^2\ d(x)$$

$$1 = C\int_0^{\infty} x^{3-1} e^{\frac{-x}{\theta^{-1}}}\ d(x) \qquad \text{(i)}$$

As we know that gamma function is

$$\overline{)a}\ b^a = \int_0^{\infty} x^{a-1}\ e^{\frac{-x}{b}}\ d(x) \qquad \text{(ii)}$$

Comparing equations (i) and (ii)

$$1 = \overline{)3}\ (\theta^{-1})^3$$

$$1 = {}^{2c}\!/_{\theta^3}$$

$$c = {}^{\theta^3}\!/_{2}$$

So

$$f(x) = \frac{\theta^3}{2} e^{-\theta x} x^2$$

Taking likelihood function

$$L(\underline{X}) = \prod_{i=1}^{n} f(x)$$

$$f(x) = \left(\frac{\theta^3}{2}\right)^n (\prod_{i=1}^{n} x)^2 \quad e^{-\theta\sum x}$$

$$f(x) = \frac{\theta^{3n}}{2^n} (\prod_{i=1}^{n} x)^2 \quad e^{-\theta\sum x}$$

At

$$H_0 : \theta = 1$$

Then we get

$$L(\omega) = \frac{1^{3n}}{2^n} (\prod_{i=1}^{n} x)^2 \quad e^{-\sum x}$$

$$L(\hat{\omega}) = \frac{1}{2^n} (\prod_{i=1}^{n} x)^2 \quad e^{-\sum x}$$

At

$$H_A : \theta \neq 1$$

$$L(x) = \frac{\theta^{3n}}{2^n} (\prod_{i=1}^{n} x)^2 \quad e^{-\theta\sum x}$$

Taking log of L.H.F

$$\log L(x) = 3n\log\theta - n\log 2 + 2\sum \log x - \theta\sum x \log e$$

$$\log L(x) = 3n\log\theta - n\log 2 + 2\sum \log x - \theta\sum x$$

Differentiate w.r.t θ

$$\frac{d\log L(x)}{d\theta} = \frac{3n}{\theta} - \sum x + 0 - 0$$

$$\frac{d\log L(x)}{d\theta} = \frac{3n}{\theta} - \sum x$$

$$\sum x = \frac{3n}{\theta}$$

$$\theta = \frac{3n}{\sum x}$$

$$\hat{\theta} = 3/\bar{x}$$

Replacing in $\hat{\theta} = {}^{3}\!/_{\bar{x}}$ L(x)

$$L(\hat{\Omega}) = \frac{\frac{3^{3n}}{\bar{x}}}{2^n}(\prod_{i=1}^{n} x)^2 \quad e^{-\frac{3}{\bar{x}}\sum x}$$

$$L(\hat{\Omega}) = \frac{3^{3n}}{2^n \bar{x}^{-3n}}(\prod_{i=1}^{n} x)^2 \quad e^{-\frac{3}{\bar{x}}\sum x}$$

$$L(\hat{\Omega}) = \frac{3^{3n}}{2^n \bar{x}^{-3n}}(\prod_{i=1}^{n} x)^2 \quad e^{-3n}$$

By likelihood ratio test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} \le k_\alpha$$

$$\frac{\frac{1}{2^n}(\prod_{i=1}^{n} x)^2 \quad e^{-n\bar{x}}}{\frac{3^{3n}}{2^n \bar{x}^{-3n}}(\prod_{i=1}^{n} x)^2 e^{-3n}} \le k_\alpha$$

$$\frac{\bar{x}^{-3n} \quad e^{-n\bar{x}}}{3^{3n} e^{-3n}} \le k_\alpha$$

$$\bar{x}^{-3n} \quad e^{-n\bar{x}} \le k_\alpha \quad 3^{3n} e^{-3n}$$

Taking log on both sides

$$3n \log \bar{x} - n\bar{x} \le \log(k_\alpha \quad 3^{3n} e^{-3n})$$

$$n(3\log \bar{x} - \overline{x}) \le \log(k_\alpha \quad 3^{3n} e^{-3n})$$

$$n(3\log \bar{x} - \overline{x}) \le \log(k_\alpha \quad 3^{3n} e^{-3n})$$

$$3\log \bar{x} - \bar{x} \quad \frac{1}{n} \le \log(k_\alpha \quad 3^{3n} e^{-3n})$$

$$-(\bar{x} - 3\log \bar{x}) \le -\frac{1}{n}\log(k_\alpha \quad 3^{3n} e^{-3n})$$

Multiply by “-1” And sign of inequality will be change

$$(\bar{x} - 3\log \bar{x}) \ge -\frac{1}{n}\log(k_\alpha \quad 3^{3n} e^{-3n})$$

Therefore $\quad C = -\frac{1}{n}\log(k_\alpha \quad 3^{3n} e^{-3n})$

$$\bar{x} - 3\log \bar{x} \ge C$$

So c is the critical region to test $H_o : \theta = 1$ Vs $H_A : \theta \ne 1$

Question no 2

Derive the likelihood ratio test based on "n" independent observation for testing $H_o:\theta=1$ Vs $H_A:\theta\neq 1$ For $f(x;\theta) = C\ e^{-\theta x}$ $\qquad x\geq 0$

Solution

As we know that

$$Area=\int_{-\infty}^{\infty} f(x)\ \ d(x)$$

$$1=C\int_{0}^{\infty} e^{-\theta x}\ \ d(x)$$

$$1=C\int_{0}^{\infty} x^{1-1}e^{\frac{-x}{\theta^{-1}}}\ \ d(x)$$

Comparing with gamma function

$$1=C\Gamma 1\ (\theta^{-1})^{1}$$

$$1=C/\theta$$

$$\theta=c$$

This is required result

So

$$f(\theta) = \theta\ e^{-\theta x} \qquad x\geq 0$$

Then likelihood function

$$L(\underline{x}) = \theta^{n}\ e^{-\theta\sum x}$$

At $H_o:\theta=1$

$$L(\hat{\omega}) = e^{-\sum x}$$

At $H_A:\theta\neq 1$

$$L(\underline{x}) = \theta^{n}\ e^{-\theta\sum x} \qquad \text{(i)}$$

Taking log on both sides

$$\log L(\underline{x}) = n\log\theta-\theta\sum x$$

Differentiate w.r.t θ

$$\frac{d\log L(x)}{d\theta} = \frac{n}{\theta}-\sum x$$

$$0 = \frac{n}{\theta}-\sum x$$

$$\sum x = \frac{n}{\theta}$$

$$\theta = \frac{n}{\sum x}$$

$$\hat{\theta} = \frac{1}{\bar{x}}$$

Replace in L(x)

$$L(\hat{\Omega}) = \left(\frac{1}{\bar{x}}\right)^n e^{-\frac{1}{\bar{x}}\sum x}$$

$$L(\hat{\Omega}) = \left(\frac{1}{\bar{x}}\right)^n e^{-n}$$

By definition of L.H.R test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} \le k_\alpha$$

$$\frac{e^{-n\bar{x}}}{\left(\frac{1}{\bar{x}}\right)^n e^{-n}} \le k_\alpha$$

$$\frac{\bar{x}^n e^{-n\bar{x}}}{e^{-n}} \le k_\alpha$$

$$\bar{x}^n e^{-n\bar{x}} \le k_\alpha e^{-n}$$

Taking log on both sides

$$n\log\bar{x} - n\bar{x} \le \log(k_\alpha e^{-n})$$

$$n(\log\bar{x} - \bar{x}) \le \log(k_\alpha e^{-n})$$

$$\log\bar{x} - \bar{x} \le \frac{1}{n}\log(k_\alpha e^{-n})$$

$$-(\bar{x} - \log\bar{x}) \le \frac{1}{n}\log(k_\alpha e^{-n})$$

Multiply by "-1"

$$\bar{x} - \log\bar{x} \ge -\frac{1}{n}\log(k_\alpha e^{-n})$$

$$\therefore C = -\frac{1}{n}\log(k_\alpha e^{-n})$$

$$\bar{x} - \log\bar{x} \ge C$$

So c is the critical region to test $H_o: \theta = 1$ vs $H_A: \theta \ne 1$.

Question no 3

A random sample of size "n" from a normal population with unknown mean and variance δ^2 is to be used to test $H_o : \mu = \mu_o$ vs $H_A : \mu \neq \mu_o$ using MLE of "μ" and δ^2.show that likelihood statistic can be written in the from.

$$\lambda = \left[\frac{1}{1+\frac{t^2}{n-1}} \right]^{\frac{n}{2}}$$

As $x \to N(\mu, \delta^2)$

$$f(x) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2\delta^2}(x-\mu)^2} \qquad -\infty \leq x \leq +\infty$$

Then taking likelihood function

$$L(x) = \underset{i=1}{\overset{n}{\pi}} f(x)$$

$$L(x) = \left(\frac{1}{\delta\sqrt{2\pi}} \right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

$$L(x) = \left(\frac{1}{\delta^2} \right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}} \right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

As $H_o : \mu = \mu_o$

$$L(\omega) = \left(\frac{1}{\delta^2} \right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}} \right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu_o)^2}$$

Taking log on both side

$$\log L(\omega) = -n \log \sqrt{2\pi} - \frac{n}{2} \log \delta^2 - \frac{1}{2\delta^2} \sum (x-\mu_0)^2$$

Partially differentiate w.r.t δ^2

$$\frac{d \log L(\omega)}{d\delta^2} = 0 - \frac{n}{2\delta^2} + \frac{1}{2\delta^4} \sum (x-\mu_0)^2$$

$$0 = -\frac{n}{2\delta^2} + \frac{1}{2\delta^4} \sum (x-\mu_0)^2$$

$$\frac{n}{2\delta^2} = \frac{1}{2\delta^4} \sum (x-\mu_0)^2$$

$$\hat{\delta}^2 = \frac{\sum (x-\mu_0)^2}{n}$$

$$\frac{1}{\hat{\delta}^2} = \frac{n}{\sum (x-\mu_0)^2}$$

Now we replace $\frac{1}{\hat{\delta}^2}$ in L(ω)then we get

$$L(\hat{\omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\mu_0)^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2\sum(x-\mu_0)^2}\sum(x-\mu_0)^2}$$

$$L(\hat{\omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\mu_0)^2}\right)^{\frac{n}{2}} e^{\frac{-n}{2}}$$

$$Now\ H_A : \mu \neq \mu_o\ \ in\ \ L.H.F\ so\ we\ \ get$$

$$L(\underline{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

Taking log on both side

$$\log L(\underline{x}) = n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{n}{2}\log\delta^2 - \frac{1}{2\delta^2}\sum(x-\mu)^2\,(-1)$$

Partially differentiate w.r.t δ^2

$$\frac{d\log L(\underline{x})}{d\delta^2} = 0 - \frac{n}{2\delta^2} + \frac{1}{2\delta^4}\sum(x-\mu)^2$$

$$\hat{\delta}^2 = \frac{\sum(x-\mu)^2}{n}$$

Again differentiate W.R.T" μ "

$$\frac{d\log L(\underline{x})}{d\mu} = 0 - 0 - \frac{2}{2\delta^2}\sum(x-\mu)^2\,(-1)$$

$$0 = \frac{\sum(x-\mu)^2}{\delta^2}$$

$$\sum(x-\mu)^2 = 0$$

$$\sum x - n\mu = 0$$

$$\hat{\mu} = \frac{\sum x}{n}$$

Now

$$L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2\sum(x-\bar{x})^2}\sum(x-\bar{x})^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

by likelihood ratio test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \lambda$$

$$\frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\mu_o)^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}} = \lambda$$

$$\left(\frac{\sum(x-\bar{x})^2}{\sum(x-\mu_o)^2}\right)^{\frac{n}{2}} = \lambda$$

$$\left(\frac{\sum(x-\bar{x})^2}{\sum(x-\bar{x})^2 + n(\bar{x}-\mu_o)^2}\right)^{\frac{n}{2}} = \lambda$$

$$\left(\frac{1}{\frac{(\sum x-\bar{x})^2}{\sum(x-\bar{x})^2} + \frac{n(\bar{x}-\mu_o)^2}{\sum(x-\bar{x})^2}}\right)^{\frac{n}{2}} = \lambda$$

$$\left(\frac{1}{1+\frac{n(\bar{x}-\mu_o)^2}{\sum(x-\bar{x})^2}}\right)^{\frac{n}{2}} = \lambda$$

$$\left(\frac{1}{1+\frac{n(\bar{x}-\mu_o)^2}{s^2(n-1)}}\right)^{\frac{n}{2}}=\lambda$$

$$\left(\frac{1}{1+\frac{1}{n-1}\frac{(\bar{x}-\mu_o)^2}{s^2/n}}\right)^{\frac{n}{2}}=\lambda$$

$t=\frac{(\bar{x}-\mu_o)}{s/\sqrt{n}}$ $t^2=\frac{(\bar{x}-\mu_o)^2}{s^2/n}$

$$\lambda=\left(\frac{1}{1+\frac{t^2}{n-1}}\right)^{\frac{n}{2}}$$ which is the required result

Question no 4

For a random sample of size "n" from normal distribution with mean μ and variance δ^2 test

$H_o:\mu=\mu_o \quad vs \quad H_A:\mu\neq\mu_o$.

As $x\rightarrow N(\mu,\delta^2)$

$$f(x)=\frac{1}{\delta\sqrt{2\pi}}e^{-\frac{1}{2\delta^2}(x-\mu)^2} \qquad -\infty\leq x\leq+\infty$$

Then taking likelihood function

$$L(x)=\left(\frac{1}{\delta\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

$$L(x)=\left(\frac{1}{\delta^2}\right)^{\frac{n}{2}}\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

$As\ H_o:\mu=\mu_o$

$$L(\omega)=\left(\frac{1}{\delta^2}\right)^{\frac{n}{2}}\left(\frac{1}{\sqrt{2\pi}}\right)^{n}e^{-\frac{1}{2\delta^2}\sum(x-\mu_o)^2}$$

Taking log on both side

$$\log L(\omega)=-n\log\sqrt{2\pi}-\frac{n}{2}\log\delta^2-\frac{1}{2\delta^2}\sum(x-\mu_0)^2$$

Partially differentiate w.r.t δ^2

$$\frac{d\log L(\omega)}{d\delta^2}=0-\frac{n}{2\delta^2}+\frac{1}{2\delta^4}\sum(x-\mu_0)^2$$

$$0=-\frac{n}{2\delta^2}+\frac{1}{2\delta^4}\sum(x-\mu_0)^2$$

$$\frac{n}{2\delta^2}=\frac{1}{2\delta^4}\sum(x-\mu_0)^2$$

$$\hat{\delta}^2=\frac{\sum(x-\mu_0)^2}{n}$$

$$\frac{1}{\hat{\delta}^2}=\frac{n}{\sum(x-\mu_0)^2}$$

Now we replace $\frac{1}{\hat{\delta}^2}$ in L(ω)then we get

$$L(\hat{\omega})=\left(\frac{1}{\sqrt{2\pi}}\right)^{n}\left(\frac{n}{\sum(x-\mu_0)^2}\right)^{\frac{n}{2}}e^{-\frac{n}{2\sum(x-\mu_0)^2}\sum(x-\mu_0)^2}$$

$$L(\hat{\omega})=\left(\frac{1}{\sqrt{2\pi}}\right)^{n}\left(\frac{n}{\sum(x-\mu_0)^2}\right)^{\frac{n}{2}}e^{\frac{-n}{2}}$$

$Now\ H_A:\mu\neq\mu_o\ in\ L.H.F\ so\ we\ get$

$$L(\underline{x})=\left(\frac{1}{\sqrt{2\pi}}\right)^{n}\left(\frac{1}{\delta^2}\right)^{\frac{n}{2}}e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

Taking log on both side

$$\log L(\underline{x})=n\log\left(\frac{1}{\sqrt{2\pi}}\right)-\frac{n}{2}\log\delta^2-\frac{1}{2\delta^2}\sum(x-\mu)^2$$

Partially differentiate w.r.t δ^2

$$\frac{d\log L(\underline{x})}{d\delta^2} = 0 - \frac{n}{2\delta^2} + \frac{1}{2\delta^4}\sum(x-\mu)^2$$

$$\hat{\delta}^2 = \frac{\sum(x-\mu)^2}{n}$$

Again differentiate W.R.T" μ "

$$\frac{d\log L(\underline{x})}{d\mu} = 0 - 0 - \frac{2}{2\delta^2}\sum(x-\mu)^2(-1)$$

$$0 = \frac{\sum(x-\mu)^2}{\delta^2}$$

$$\sum(x-\mu)^2 = 0$$

$$\sum x - n\mu = 0$$

$$\hat{\mu} = \frac{\sum x}{n}$$

Now

$$L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2\sum(x-\bar{x})^2}\sum(x-\bar{x})^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

by likelihood ratio test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} \le k_\alpha$$

$$\frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\mu_o)^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{n}{\sum(x-\bar{x})^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}} \le k_\alpha$$

$$\left(\frac{\sum(x-\bar{x})^2}{\sum(x-\mu_o)^2}\right)^{\frac{n}{2}} \le k_\alpha$$

Let $\sum(x-\mu_o)^2 = \sum(x-\bar{x}+\bar{x}-\mu_o)^2$

$$\sum(x-\bar{x})^2 + n(\bar{x}-\mu_o)^2 - 2(x-\bar{x})(\bar{x}-\mu_o)$$

$$\sum(x-\bar{x})^2 + n(\bar{x}-\mu_o)^2$$

$$\left(\frac{\sum(x-\bar{x})^2}{\sum(x-\bar{x})^2 + n(\bar{x}-\mu_o)^2}\right)^{\frac{n}{2}} \le k_\alpha$$

$$\left(\frac{1}{\frac{(\sum x-\bar{x})^2}{\sum(x-\bar{x})^2} + \frac{n(\bar{x}-\mu_o)^2}{\sum(x-\bar{x})^2}}\right)^{\frac{n}{2}} \le k_\alpha$$

$$\left(\frac{1}{1+\frac{n(\bar{x}-\mu_o)^2}{\sum(x-\bar{x})^2}}\right)^{\frac{n}{2}} \le k_a$$

$$\left(\frac{1}{1+\frac{n(\bar{x}-\mu_o)^2}{s^2(n-1)}}\right)^{\frac{n}{2}} \le k_a$$

$$\left(\frac{1}{1+\frac{1}{n-1}\frac{(\bar{x}-\mu_o)^2}{s^2/n}}\right)^{\frac{n}{2}} \le k_a$$

$t = \dfrac{(\bar{x}-\mu_o)}{s/\sqrt{n}}$ $t^2 = \dfrac{(\bar{x}-\mu_o)^2}{s^2/n}$

$$\left(\frac{1}{1+\frac{t^2}{n-1}}\right)^{\frac{n}{2}} \le k_a$$

$$\left(1+\frac{t^2}{n-1}\right)^{-\frac{n}{2}} \le k_a$$

Taking power -2/n

$$1+\frac{t^2}{n-1} \le k_a^{-\frac{2}{n}}$$

$$\frac{t^2}{n-1} \le k_a^{-\frac{2}{n}} - 1$$

$$t^2 \le (k_a^{-\frac{2}{n}} - 1)(n-1)$$

$$|t| \le \sqrt{(k_a^{-\frac{2}{n}} - 1)}\sqrt{(n-1)}$$

Is a likelihood ration test to $H_o : \mu = \mu_o \quad vs \quad H_A : \mu \neq \mu_o$.

Question no 5

If $x \to N(\mu, \delta^2)$ where δ^2 is known. Test $H_o : \mu = \mu_o \quad vs \quad H_A : \mu \neq \mu_o$

As $x \to N(\mu, \delta^2)$

$$f(x) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{1}{2\delta^2}(x-\mu)^2} \qquad -\infty \le x \le +\infty$$

Taking likelihood function

$$L(x) = \left(\frac{1}{\delta\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

$$L(x) = \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2}$$

$$As\ H_o : \mu = \mu_o$$

$$L(\hat{\omega}) = \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu_o)^2} \qquad \rightarrow (1)$$

As δ^2 is known there is no need to estimate the parameter.

$$Now\ H_A : \mu \neq \mu_o$$

$$L(\underline{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2} \qquad \rightarrow (2)$$

Taking log on both side

$$\log L(\underline{x}) = n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{n}{2}\log\delta^2 - \frac{1}{2\delta^2}\sum(x-\mu)$$

differentiate W.R.T" μ "

$$\frac{d\log L(\underline{x})}{d\mu} = 0 - 0 - \frac{2}{2\delta^2}\sum(x-\mu)$$

$$0 = \frac{\sum(x-\mu)}{\delta^2}$$

$$\sum x - n\mu = 0$$

$$\sum x = n\mu$$

$$\mu = \bar{x}$$

Put the value equation 2

$$L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\delta^2}\sum(x-\bar{x})^2}$$

By likelihood ratio test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} \leq k_\alpha$$

$$\frac{\left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu_o)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\delta^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\delta^2}\sum(x-\bar{x})^2}} \leq k_a$$

$$e^{-\frac{1}{2\delta^2}\sum(x-\mu_o)^2+\frac{1}{2\delta^2}\sum(x-\bar{x})^2} \leq k_a$$

Take log on both side

$$-\frac{1}{2\delta^2}\sum(x-\mu_o)^2+\frac{1}{2\delta^2}\sum(x-\bar{x})^2 \leq \log k_a$$

$$-\frac{1}{2\delta^2}\left(\sum x^2-n\mu_o{}^2-2\mu_o\sum x-\sum x^2-n\bar{x}^2+2\bar{x}\sum x\right) \leq \log k_a$$

$$-\frac{1}{2\delta^2}\left(n\mu_o{}^2-2\mu_o\sum x-n\bar{x}^2+2n\bar{x}^2\right) \leq \log k_a$$

$$-\frac{1}{2\delta^2}\left(n\mu_o{}^2-2\mu_o n\bar{x}+n\bar{x}^2\right) \leq \log k_a$$

$$-\frac{1}{2\delta^2}\left(n(\bar{x}-\mu_o)^2\right) \leq \log k_a$$

$$-n(\bar{x}-\mu_o)^2 \leq 2\delta^2\log k_a$$

Multiply “-1” and reverse inequality

$$n(\bar{x}-\mu_o)^2 \geq -2\delta^2\log k_a$$

$$(\bar{x}-\mu_o)^2 \geq -\frac{2\delta^2}{n}\log k_a$$

Taking square root on both side

$$\left|\bar{x}-\mu_o\right| \geq -\sqrt{\frac{2\delta^2}{n}\log k_a}$$

$$\therefore\ c = -\sqrt{\frac{2\delta^2}{n}\log k_a}$$

$$\left|\bar{x}-\mu_o\right| \geq C$$

Question no 6

Let f(x) be a random sample distribution with variance “1” and unknown mean $"\mu"$ .let the random sample consists of “n” observations find the critical region of likelihood ratio test of size “a” for test $H_o : \mu = 3 \quad vs \quad H_1 : \mu \neq 3$

As $x \to N(\mu\,,\,1)$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} \qquad -\infty \leq x \leq +\infty$$

Take likelihood function

$$L(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x-\mu)^2}$$

$As \quad H_o : \mu = 3$

$$L(\hat{\omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x-3)^2}$$

$Now \quad H_1 : \mu \neq 3$

$$L(\underline{x}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum(x-\mu)^2} \qquad \to (1)$$

Taking log on both side

$$\log L(\underline{x}) = n \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\Sigma(x-\mu)$$

differentiate W.R.T” μ ”

$$\frac{d \log L(\underline{x})}{d\mu} = 0 - \frac{2}{2}\sum(x-\mu)(-1)$$

$$0 = \frac{\sum(x-\mu)}{\delta^2}$$

$$\sum x - n\mu = 0$$

$$\sum x = n\mu$$

$$\hat{\mu} = \overline{x}$$

Now put in equation (1)

$$L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum (x-\bar{x})^2}$$

By likelihood ratio test

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} \leq k_\alpha$$

$$\frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum (x-3)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\delta^2}\sum (x-\bar{x})^2}} \leq k_a$$

$$\frac{e^{-\frac{1}{2}\sum (x-3)^2}}{e^{-\frac{1}{2\delta^2}\sum (x-\bar{x})^2}} \leq k_a$$

$$e^{-\frac{1}{2}\sum (x-3)^2 - \frac{1}{2\delta^2}\sum (x-\bar{x})^2} \leq k_a$$

Taking log on both side

$$-\frac{1}{2}\sum (x-3)^2 - \frac{1}{2\delta^2}\sum (x-\bar{x})^2 \leq \log k_a$$

$$-\frac{1}{2}\left(\sum x^2 + 9n - 6\sum x - \sum x^2 - n\bar{x}^2 + 2\bar{x}n\bar{x}\right) \leq \log k_a$$

$$-\frac{1}{2}\left(9n - 6n\bar{x} + n\bar{x}^2\right) \leq \log k_a$$

$$-\frac{1}{2}\left(n(\bar{x}-3)^2\right) \leq \log k_a$$

$$-n(\bar{x}-3)^2 \leq 2\log k_a$$

$$(\bar{x}-3)^2 \leq -\frac{2}{n}\log k_a$$

$$\left|\bar{x}-3\right| \leq -\sqrt{\frac{2}{n}\log k_a}$$

$$\therefore C = \sqrt{\frac{2}{n}\log k_a}$$

$$\left|\bar{x}-3\right| \leq C$$

This is the required result.

Properties Of Likelihood Ratio Test:

There are some important properties of Likelihood Ratio Test as given below:

1) The ratio of $'\lambda'$ satisfied $0 < \lambda < 1 \; i.e \; \lambda \geq 0$.
2) Parameter $'\theta'$ cannot be vector valued the denominator.
3) Denominator of $'\lambda'$ is the likelihood function as the ML estimator.
4) The sample $x_1, x_2, x_3, \ldots\ldots, x_n$ is the random sample from the p.d.f $f(x;\theta)$.
5) It is used to find best critical region.

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